

ORIENTABILITY FOR GAUGE THEORIES ON CALABI-YAU MANIFOLDS

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ABSTRACT. We study orientability issues of moduli spaces from gauge theories on Calabi-Yau manifolds. Our results generalize and strengthen those for Donaldson-Thomas theory on Calabi-Yau manifolds of dimensions 3 and 4. We also prove a corresponding result in the relative situation which is relevant to the gluing formula in DT theory.

MSC 2010: 14N35 14J32 53C07 81T13

1. INTRODUCTION

Donaldson invariants count anti-self-dual connections on closed oriented 4-manifolds [17]. The definition requires an orientability result proved by Donaldson in [19]. Indeed, Donaldson theory fits into the 3-dimensional TQFT structure in the sense of Atiyah [2]. In particular, relative Donaldson invariants for $(X, Y = \partial X)$ take values in the instanton Chern-Simons-Floer (co)homology $HF_{CS}^*(Y)$ [20, 52]. The Euler characteristic of $HF_{CS}^*(Y)$ is the Casson invariant which counts flat connections on a closed 3-manifold Y .

As was proposed by Donaldson and Thomas [22], we are interested in the *complexification* of the above theory. Namely, we consider holomorphic vector bundles (or general coherent sheaves) over Calabi-Yau manifolds [53]. The complex analogs of (i) Donaldson invariants, (ii) Chern-Simons-Floer (co)homology $HF_{CS}^*(Y)$, and (iii) Casson invariants are (i) DT_4 invariants, (ii) DT_3 (co)homology $H_{DT_3}^*(Y)$, and (iii) DT_3 invariants.

As a complexification of Casson invariants, Thomas defined Donaldson-Thomas invariants for Calabi-Yau 3-folds [48]. DT_3 invariants for ideal sheaves of curves are related to many other interesting subjects including Gopakumar-Vafa conjecture on BPS numbers in string theory [25, 27, 34] and MNOP conjecture [38, 39, 40, 44] which relates DT_3 invariants and Gromov-Witten invariants. The generalization of DT_3 invariants to count strictly semi-stable sheaves is due to Joyce and Song [33] using Behrend's result [5]. Kontsevich and Soibelman proposed generalized as well as motivic DT theory for Calabi-Yau 3-categories [35], which was later studied by Behrend, Bryan and Szendrői [6] for Hilbert schemes of points. The wall-crossing formula [35, 33] is an important structure for Bridgeland's stability condition [11] and Pandharipande-Thomas invariants [45, 49].

As a complexification of Chern-Simons-Floer theory, Brav, Bussi, Dupont, Joyce and Szendrői [9], Kiem and Li [34] recently defined a cohomology theory on Calabi-Yau 3-folds whose Euler characteristic is the DT_3 invariant. The point is that moduli spaces of simple sheaves on Calabi-Yau 3-folds are locally critical points of holomorphic functions [10, 33], and we could consider perverse sheaves of vanishing cycles of these functions. They glued these local perverse sheaves and defined its hypercohomology as DT_3 cohomology. In general, gluing these perverse sheaves requires a square root of the determinant line bundle of the moduli space. Nekrasov and Okounkov proved its existence in [43]. The square root is called an orientation data if it is furthermore compatible with wall-crossing (or Hall algebra structure) [35] whose existence was proved by Hua on simply-connected torsion-free CY_3 [28].

As a complexification of Donaldson theory, Borisov and Joyce [7] and the authors [13, 14] developed DT_4 invariants (or 'holomorphic Donaldson invariants') which count stable sheaves on Calabi-Yau 4-folds. To define the invariants, we need an orientability result, which was solved by the authors in [14] for Calabi-Yau 4-fold X which satisfies $H^{odd}(X, \mathbb{Z}) = 0$ (for instance, complete intersections in smooth toric varieties satisfy this condition).

In this paper, we show that all these orientability results have their origin in spin geometry [1, 36], and then generalize and strengthen them to Calabi-Yau manifolds of any dimension.

Let us start with a compact spin manifold X of even dimension and a (Hermitian) complex vector bundle $(E, h) \rightarrow X$. Given an unitary connection A on E , one can define the twisted Dirac operator

$$\not{D}_{A^* \otimes A} : \Gamma(\mathcal{S}_{\mathbb{C}}^+(X) \otimes \text{End} E) \rightarrow \Gamma(\mathcal{S}_{\mathbb{C}}^-(X) \otimes \text{End} E)$$

following Theorem 13.10 of [36]. $[\ker(\mathcal{D}_{A^* \otimes A}) - \operatorname{coker}(\mathcal{D}_{A^* \otimes A})]$ exists as an element in the K -theory $K(pt)$ of one point, and there is a family version of the above construction over the space $\tilde{\mathcal{B}}_X$ of gauge equivalent classes of framed unitary connections on E . The index bundle

$$\operatorname{Ind}(\mathbb{P}_{\operatorname{End} E}) \in K(\tilde{\mathcal{B}}_X),$$

exists [4] whose determinant $\mathcal{L}_{\mathbb{C}} = \det(\operatorname{Ind}(\mathbb{P}_{\operatorname{End} E}))$ is a complex line bundle over $\tilde{\mathcal{B}}_X$. This determinant line bundle has some remarkable properties depending on the dimension of X . To explain that, we first recall the following standard facts about spin geometry (see also Theorem 5.1).

Lemma 1.1. ([1], [36]) *Let $\mathcal{S}_{\mathbb{C}}^{\pm}(X)$ be the complex spinor bundles of an even dimensional spin manifold X . Then*

- (1) *if $\dim X = 8k$, there exists real spinor bundles $\mathcal{S}^{\pm}(X)$ such that $\mathcal{S}_{\mathbb{C}}^{\pm}(X) = \mathcal{S}^{\pm}(X) \otimes_{\mathbb{R}} \mathbb{C}$;*
- (2) *if $\dim X = 4k + 2$, $\mathcal{S}_{\mathbb{C}}^{+}(X) \cong (\mathcal{S}_{\mathbb{C}}^{-}(X))^*$ as Clifford bundles.*

From this lemma, we can obtain corresponding structures on $\mathcal{L}_{\mathbb{C}}$, i.e.

- (1) *if $\dim X = 8k$, there exists a real line bundle $\mathcal{L}_{\mathbb{R}}$ such that $\mathcal{L}_{\mathbb{C}} \cong \mathcal{L}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. In other words, there exists a non-degenerate quadratic form Q on $\mathcal{L}_{\mathbb{C}}$ with*

$$Q : \mathcal{L}_{\mathbb{C}} \otimes \mathcal{L}_{\mathbb{C}} \cong \mathbb{C} \times \tilde{\mathcal{B}}_X.$$

- (2) *If $\dim X = 4k + 2$, the extended determinant line bundle $\mathcal{L}_{\mathbb{C}} \rightarrow \tilde{\mathcal{B}}_X \times \tilde{\mathcal{B}}_X$ (see Section 3) satisfies¹*

$$\sigma^* \mathcal{L}_{\mathbb{C}} \cong \mathcal{L}_{\mathbb{C}},$$

where

$$\sigma : \tilde{\mathcal{B}}_X \times \tilde{\mathcal{B}}_X \rightarrow \tilde{\mathcal{B}}_X \times \tilde{\mathcal{B}}_X,$$

$$\sigma([A_1], [A_2]) = ([A_2], [A_1]).$$

Furthermore, we will show the following orientability result.

Theorem 1.2. (Theorem 2.1, Theorem 3.1)

Let X be a compact spin manifold of even dimension, and (E, h) be a Hermitian complex vector bundle. Then

- (1) *if $\dim X = 8k$, the structure group of $(\mathcal{L}_{\mathbb{C}}, Q)$ can be reduced to $SO(1, \mathbb{C})$, i.e. the corresponding real line bundle $\mathcal{L}_{\mathbb{R}}$ is trivial, provided that $H_{\text{odd}}(X, \mathbb{Z}) = 0$;*
- (2) *if $\dim X = 4k + 2$, $\mathcal{L}_{\mathbb{C}}$ has natural² choices of square roots parametrized by $\operatorname{Hom}(H^{\text{odd}}(X, \mathbb{Z}), \mathbb{Z}_2)$.*

On Calabi-Yau manifolds, $\mathcal{S}_{\mathbb{C}}(X) = \bigwedge^{0,*}(X)$ and $\mathcal{D} = \bar{\partial}$, the above result gives an orientability for (coarse) moduli spaces of simple holomorphic bundles. By a machinery (heavily used by Joyce-Song [33]) called Seidel-Thomas twist [47], we can extend it to moduli spaces of simple coherent sheaves.

Theorem 1.3. (Theorem 2.2, Theorem 3.2)

Let X be a projective Calabi-Yau n -fold with $\operatorname{Hol}(X) = SU(n)$, \mathcal{M}_X be a coarse moduli space of simple sheaves with fixed Chern classes³, and we denote its determinant line bundle by $\mathcal{L}_{\mathcal{M}_X}$. Then, we have

- (1) *if $n = 4k$, structure group of $(\mathcal{L}_{\mathcal{M}_X}, Q_{\operatorname{Serre}})$ can be reduced to $SO(1, \mathbb{C})$, when $H_{\text{odd}}(X, \mathbb{Z}) = 0$*
- (2) *if $n = 4k + 2$, the structure group of $(\mathcal{L}_{\mathcal{M}_X}, Q_{\operatorname{Serre}})$ is canonically reduced to $SO(1, \mathbb{C})$ ⁴;*
- (3) *if n is odd, each element in $\operatorname{Hom}(H^{\text{odd}}(X, \mathbb{Z}), \mathbb{Z}_2)$ determines an (algebraic) square root of $\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{\text{red}}}$ over the reduced scheme $\mathcal{M}_X^{\text{red}}$, when \mathcal{M}_X is a proper scheme.*

This result in fact fits into the work of Borisov and Joyce on the definition of orientations for derived schemes with shifted symplectic structures (see Definition 2.11 of [7]) and Joyce's definition of orientations for d-critical loci [31] (used to categorify DT_3 invariants). In general, derived moduli schemes of simple sheaves on Calabi-Yau n -folds are expected to have $(2 - n)$ -shifted symplectic structures in the sense of Pantev, Töen, Vaquié and Vezzosi [46]. When n is even, there is a canonical isomorphism

$$Q_{\operatorname{Serre}} : \mathcal{L}_{\mathcal{M}_X} \otimes \mathcal{L}_{\mathcal{M}_X} \cong \mathcal{O}_{\mathcal{M}_X}$$

¹This follows from a brilliant idea due to Maulik, Nekrasov and Okounkov [43].

²See Theorem 3.1 for the precise meaning.

³We endow it with the induced complex analytic topology as page 54 of [33]. One could also impose the Gieseker stability condition to get a projective scheme as the moduli space [30].

⁴This is first observed by Borisov and Joyce [7].

and the orientability issue is to find a square root of this isomorphism [7]. When n is odd, the orientability issue is to find a square root of $\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}}$ over the reduced scheme \mathcal{M}_X^{red} [31].

Along this line, we also prove an orientability result for the relative situation where we have Calabi-Yau manifolds as anti-canonical divisors of even dimensional projective manifolds. This will be useful in the relative DT_4 theory [15] (which is part of the complexification of Donaldson-Floer TQFT theory on 4-3 dimensional manifolds).

Theorem 1.4. (*Weak relative orientability, Theorem 4.1*)

Let Y be a smooth anti-canonical divisor in a projective $2n$ -fold X with $\text{Tor}(H_*(X, \mathbb{Z})) = 0$, $E \rightarrow X$ be a complex vector bundle with structure group $SU(N)$, where $N \gg 0$. Let \mathcal{M}_X be a coarse moduli scheme of simple holomorphic structures on E , which has a well-defined restriction morphism

$$r : \mathcal{M}_X \rightarrow \mathcal{M}_Y,$$

to a proper coarse moduli scheme of simple bundles on Y with fixed Chern classes.

Then there exists an algebraic square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$ of $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}$ such that

$$c_1(\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}}) = r^*c_1((\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}),$$

where $\mathcal{L}_{\mathcal{M}_X}$ (resp. $\mathcal{L}_{\mathcal{M}_Y}$) is the determinant line bundle of \mathcal{M}_X (resp. \mathcal{M}_Y).

Given such a restriction morphism r , \mathcal{M}_X is expected to be a Lagrangian (see Calaque [12]) of the derived scheme \mathcal{M}_Y with $(3 - 2n)$ -shifted symplectic structure in the sense of Pantev, Töen, Vaquié and Vezzosi [46]. Then there is a canonical isomorphism

$$(\mathcal{L}_{\mathcal{M}_X})^{\otimes 2} \cong r^*\mathcal{L}_{\mathcal{M}_Y}$$

between determinant line bundles (it is verified directly in Lemma 4.2). The orientability issue in this relative case is to find a square root of this isomorphism (see Definition 4.4), which is partially verified by the above weak relative orientability result and Proposition 4.6.

Content of the paper: In section 2, we study the orientability issue for moduli spaces of simple sheaves on Calabi-Yau manifolds of even dimensions. We first prove a general result for spin manifolds of $8k$ dimensions and then apply it to the case of Calabi-Yau even-folds. Then we discuss its relation to the work of Borisov and Joyce [7] on the definition of orientations for derived schemes with shifted symplectic structures [46]. In section 3, we study the orientability issue for moduli spaces of simple sheaves on Calabi-Yau manifolds of odd dimensions (and corresponding results on spin manifolds of $(4k + 2)$ dimensions). The orientability result fits into Joyce's definition of orientations for d-critical loci [31] (used to categorify DT_3 invariants). In section 4, we discuss the orientability issue for the relative situation. We define relative orientations for restriction morphisms and partially verify their existences. In the appendix, we list some useful facts on spin geometry, gauge theory and Seidel-Thomas twists.

Acknowledgement: The first author expresses his deep gratitude to Professor Simon Donaldson, Dominic Joyce and Tony Pantev for many useful discussions. We thank Zheng Hua for various helpful discussions and comments. The work of the second author was substantially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK401411 and CUHK14302714).

2. ORIENTABILITY FOR EVEN DIMENSIONAL CALABI-YAU

We fix a compact spin manifold X of even dimension and a (Hermitian) complex vector bundle $(E, h) \rightarrow X$. Given an unitary connection A on E , we define the twisted Dirac operator

$$\not{D}_{A^* \otimes A} : \Gamma(\not{S}_C^+(X) \otimes \text{End} E) \rightarrow \Gamma(\not{S}_C^-(X) \otimes \text{End} E)$$

following Theorem 13.10 of [36]. $[\ker(\not{D}_{A^* \otimes A}) - \text{coker}(\not{D}_{A^* \otimes A})]$ exists as an element in the K -theory $K(pt)$ of one point, and there is a family version of the above construction as follows⁵.

Let \mathcal{A} be the space of all unitary connections on (E, h) , and \mathcal{G} be the group of unitary gauge transformations. We denote the $U(r)$ -principal bundle (of frames) of E by P , fix a base point $x_0 \in X$ and introduce the space

$$\tilde{\mathcal{B}}_X = \mathcal{A} \times_{\mathcal{G}} P_{x_0} \quad (= \tilde{\mathcal{B}}_{E, X})$$

of gauge equivalent classes of framed connections. Equivalently, $\tilde{\mathcal{B}}_X = \mathcal{A}/\mathcal{G}_0$, where $\mathcal{G}_0 \triangleleft \mathcal{G}$ is the subgroup of gauge transformations which fix the fiber P_{x_0} . As \mathcal{G}_0 acts freely on \mathcal{A} , $\tilde{\mathcal{B}}_X$ (with

⁵More details are explained in the appendix.

suitable Sobolev structure) has a Banach manifold structure whose weak homotopy type will not depend on the chosen Sobolev structures (Proposition 5.1.4 [21]).

Meanwhile, there exists a universal bundle $\mathcal{E} = \mathcal{A} \times_{\mathcal{G}_0} E$ over $\tilde{\mathcal{B}}_X \times X$ (trivialized on $\tilde{\mathcal{B}}_X \times \{x_0\}$), which carries a universal family of framed connections. We then couple the Dirac operator \mathcal{D} on X with the connection on \mathcal{E} and there is an index bundle

$$\text{Ind}(\mathcal{D}_{\text{End}\mathcal{E}}) \in K(\tilde{\mathcal{B}}_X),$$

which satisfies $\text{Ind}(\mathcal{D}_{\text{End}\mathcal{E}})|_{[A]} = \ker(\mathcal{D}_{A^* \otimes A}) - \text{coker}(\mathcal{D}_{A^* \otimes A}) \in K(pt)$ [4]. The determinant $\mathcal{L}_{\mathbb{C}} = \det(\text{Ind}(\mathcal{D}_{\text{End}\mathcal{E}}))$ of $\text{Ind}(\mathcal{D}_{\text{End}\mathcal{E}})$ exists as a complex line bundle over $\tilde{\mathcal{B}}_X$. Meanwhile, the $U(r)$ -action on $\tilde{\mathcal{B}}_X$ which changes framing at P_{x_0} naturally extends to the line bundle $\mathcal{L}_{\mathbb{C}} \rightarrow \tilde{\mathcal{B}}_X$.

If the spin manifold X is of real dimension $8k$, the complex spinor bundle $\mathcal{S}_{\mathbb{C}}^{\pm}(X)$ is the complexification of real spinor bundle $\mathcal{S}^{\pm}(X)$, i.e. $\mathcal{S}_{\mathbb{C}}^{\pm}(X) = \mathcal{S}^{\pm}(X) \otimes_{\mathbb{R}} \mathbb{C}$ (see page 99 of [36] or Theorem 5.1). Then $\mathcal{S}_{\mathbb{C}}^{\pm}(X) \otimes_{\mathbb{C}} \text{End}E = (\mathcal{S}^{\pm}(X) \otimes_{\mathbb{R}} \mathfrak{g}_E) \otimes_{\mathbb{R}} \mathbb{C}$ and the corresponding twisted Dirac operator is the complexification of the real one. Thus the determinant line bundle $\mathcal{L}_{\mathbb{C}}$ is the complexification of a real determinant line bundle $\mathcal{L}_{\mathbb{R}} = \det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_E}))$ for twisted Dirac operators of type

$$\mathcal{D}_{A^* \otimes A} : \Gamma(\mathcal{S}^+(X) \otimes_{\mathbb{R}} \mathfrak{g}_E) \rightarrow \Gamma(\mathcal{S}^-(X) \otimes_{\mathbb{R}} \mathfrak{g}_E).$$

This then defines a non-degenerate quadratic form Q on $\mathcal{L}_{\mathbb{C}}$ and gives a trivialization

$$Q : \mathcal{L}_{\mathbb{C}} \otimes \mathcal{L}_{\mathbb{C}} \cong \mathbb{C} \times \tilde{\mathcal{B}}_X.$$

Theorem 2.1. *For any compact spin manifold X of real dimension $8k$ with $H_{\text{odd}}(X, \mathbb{Z}) = 0$, and a Hermitian vector bundle $E \rightarrow X$, the structure group of $(\mathcal{L}_{\mathbb{C}}, Q)$ can be reduced to $SO(1, \mathbb{C})$, i.e. the corresponding real line bundle $\mathcal{L}_{\mathbb{R}}$ of $(\mathcal{L}_{\mathbb{C}}, Q)$ is trivial.*

Proof. Following the approach by Donaldson [19], [21], by considering $E' = E \oplus (\det E)^{-1} \oplus \mathbb{C}^p$, we have a stabilization map

$$s : \tilde{\mathcal{B}}_{E, X} \rightarrow \tilde{\mathcal{B}}_{E', X}, \quad s(A) = A \oplus (\det(A))^* \oplus \theta$$

where θ is the rank p product connection. When a $SU(N)$ connection on E' decomposes as $A \oplus (\det(A))^* \oplus \theta$, there is a decomposition of the adjoint bundle

$$\mathfrak{g}_{E'} = \mathfrak{g}_E \oplus V \oplus \mathfrak{g}_{\mathbb{C}^p}, \quad V \triangleq ((\det E) \otimes E) \oplus ((\mathbb{C}^p)^* \otimes E) \oplus ((\det E) \otimes \mathbb{C}^p).$$

The index of any operator coupled with this bundle (with connection) is a sum of corresponding terms. In the obvious notations, we have

$$s^*(\det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_{E'}}))) = \det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_E})) \otimes \det(\text{Ind}(\mathcal{D}_V)) \otimes \det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_{\mathbb{C}^p}})).$$

As V is complex, $\det(\text{Ind}(\mathcal{D}_V))$ has a canonical orientation. $\det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_{\mathbb{C}^p}}))$ is also trivial as \mathbb{C}^p is a product bundle. Then $s^*(\det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_{E'}}))) \cong \det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_E}))$.

Then we are left to show $\det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_E}))$ is trivial for a $SU(N)$ complex vector bundle E on X with $N \gg 0$. Analogs to Theorem 10.14 of [14], we apply the Federer spectral sequence [41],

$$E_2^{p,q} \cong H^p(X, \pi_{p+q}(BSU(N))) \Rightarrow \pi_q(\text{Map}_E(X, BSU(N))).$$

For $N \gg 0$, we get $\pi_1(\text{Map}_E(X, BSU(N))) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z})$, which vanishes by the assumption. From Atiyah-Bott (Proposition 2.4 [3]), we have a homotopy equivalence

$$B\mathcal{G} \simeq \text{Map}_E(X, BSU(N)).$$

Then

$$\pi_1(\tilde{\mathcal{B}}_X) \cong \pi_1((\mathcal{A} \times SU(N))/\mathcal{G}) \cong \pi_0(\mathcal{G}) \cong \pi_1(\text{Map}_E(X, BSU(N))) = 0.$$

Then any real line bundle over $\tilde{\mathcal{B}}_X$ (including $\mathcal{L}_{\mathbb{R}} = \det(\text{Ind}(\mathcal{D}_{\mathfrak{g}_E}))$) is trivial. \square

We fix a Calabi-Yau $4n$ -fold X , and denote the determinant line bundle of a coarse moduli space \mathcal{M}_X of simple sheaves by $\mathcal{L}_{\mathcal{M}_X}$ with $\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{F}} \cong \det(\text{Ext}^{\text{odd}}(\mathcal{F}, \mathcal{F})) \otimes \det(\text{Ext}^{\text{even}}(\mathcal{F}, \mathcal{F}))^{-1}$. The Serre duality pairing defines a non-degenerate quadratic form Q_{Serre} on $\mathcal{L}_{\mathcal{M}_X}$ and gives a trivialization

$$Q_{\text{Serre}} : \mathcal{L}_{\mathcal{M}_X} \otimes \mathcal{L}_{\mathcal{M}_X} \cong \mathcal{O}_{\mathcal{M}_X}.$$

Theorem 2.2. *Let X be a projective Calabi-Yau $4n$ -fold with $H_{\text{odd}}(X, \mathbb{Z}) = 0$, \mathcal{M}_X be a coarse moduli space of simple sheaves with fixed Chern classes.*

Then the structure group of $(\mathcal{L}_{\mathcal{M}_X}, Q_{\text{Serre}})$ can be reduced to $SO(1, \mathbb{C})$, i.e. the corresponding real line bundle $\mathcal{L}_{\mathbb{R}}$ of $(\mathcal{L}_{\mathcal{M}_X}, Q_{\text{Serre}})$ is trivial. In particular, $\mathcal{L}_{\mathcal{M}_X} \cong \mathcal{O}_{\mathcal{M}_X}$.

Proof. By the work of Joyce-Song [33] (see Corollary 5.11 in the appendix), we are reduced to consider the case when \mathcal{M}_X is a coarse moduli space of simple holomorphic structures on a complex bundle E of rank r .

Let $\mathcal{A}^* \subseteq \mathcal{A}$ be the subspace of irreducible unitary connections on (E, h) , $\tilde{\mathcal{B}}_X^* = \mathcal{A}^*/\mathcal{G}_0 \subseteq \tilde{\mathcal{B}}_X$ be the open subset of irreducible framed connections whose complement is of infinite codimension (page 181 of [21]). As Section 9.1 of [33], we introduce $\mathcal{A}_{si}^{(0,1)}$ to be the space of simple $(0, 1)$ -connections on E . There is a group \mathcal{G}^c of complex gauge transformations acting on $\mathcal{A}_{si}^{(0,1)}$ with stabilizer $\mathbb{C}^* \cdot Id_E$. The subgroup \mathcal{G}_0^c which preserves a fiber E_{x_0} then acts freely on $\mathcal{A}_{si}^{(0,1)}$, and $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_0^c$ is a Banach complex manifold (with suitable Banach completions, see [33]). Via the Hermitian metric h , $\mathcal{A}_{si}^{(0,1)} \cong \mathcal{A}^*$, we then have an embedding $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_0^c \subseteq \tilde{\mathcal{B}}_X^*$. There is also a forgetful map (similar to (5.1.3) in [21])

$$\beta : \mathcal{A}_{si}^{(0,1)}/\mathcal{G}_0^c \rightarrow \mathcal{A}_{si}^{(0,1)}/\mathcal{G}_{red}^c,$$

which is a principal $PGL(r, \mathbb{C})$ -bundle, where $\mathcal{G}_{red}^c = \mathcal{G}^c/\mathbb{C}^*$ and $\mathbb{C}^* \subseteq \mathcal{G}^c$ is the subgroup of multiples of the identity map (see page 133 of [33]).

Our coarse moduli space \mathcal{M}_X of simple holomorphic bundles then sits inside $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_{red}^c$ as component(s) of integrable connections. As Calabi-Yau $4n$ -folds are spin manifolds of dimensions $8m$, there is a quadratic line bundle $(\mathcal{L}_\mathbb{C}, Q)$ on $\tilde{\mathcal{B}}_X$ from the previous discussion. Its pull-back to $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_0^c$ via the imbedding

$$\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_0^c \subseteq \tilde{\mathcal{B}}_X^* \subseteq \tilde{\mathcal{B}}_X$$

is $GL(r, \mathbb{C})$ invariant (changing the framing at E_{x_0}). So it descends to a quadratic line bundle over $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_{red}^c$ (via β) (see also 5.4.2 of [21]). On Calabi-Yau manifolds X 's, $\mathcal{S}_\mathbb{C}(X) = \bigwedge^{0,*}(X)$ and $\mathcal{D} = \bar{\mathcal{D}}$, the descended quadratic line bundle (over $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_{red}^c$) pulls back to the determinant line bundle $\mathcal{L}_{\mathcal{M}_X}$ with Q_{Serre} over \mathcal{M}_X . By Theorem 2.1, the real line bundle $\mathcal{L}_\mathbb{R}$ associated to $(\mathcal{L}_\mathbb{C}, Q)$ is trivial over $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_0^c$. As the fiber of the map β is connected, $\mathcal{L}_\mathbb{R}$ descends to a trivial line bundle over $\mathcal{A}_{si}^{(0,1)}/\mathcal{G}_{red}^c$, in particular over \mathcal{M}_X . \square

Remark 2.3. $H_{odd}(X, \mathbb{Z}) = 0$ holds true for complete intersections X 's in smooth toric varieties [24].

Given a Calabi-Yau $2n$ -fold X , and a coarse moduli space \mathcal{M}_X of simple sheaves, the Serre duality pairing gives a non-degenerate quadratic form on the determinant line bundle $\mathcal{L}_{\mathcal{M}_X}$, which defines an isomorphism

$$(1) \quad \mathcal{L}_{\mathcal{M}_X} \otimes \mathcal{L}_{\mathcal{M}_X} \cong \mathcal{O}_{\mathcal{M}_X}.$$

The above Theorem 2.2 in fact shows that we can find an isomorphism $\mathcal{L}_{\mathcal{M}_X} \cong \mathcal{O}_{\mathcal{M}_X}$ whose square is the above given isomorphism.

This fits into the work of Borisov and Joyce on the orientation of derived schemes with k -shifted symplectic structure for even $k \leq 0$ (see Definition 2.11 of [7]). In general, derived moduli schemes of simple sheaves on Calabi-Yau n -folds are expected to have $(2-n)$ -shifted symplectic structures in the sense of Pantev, Töen, Vaquié and Vezzosi [46]. When n is even, there is a canonical isomorphism like (1). The orientability issue in this case is to find a square root of this isomorphism (see [7] for more details).

The above Theorem 2.2 partially solves this issue for Calabi-Yau $4n$ -folds (which correspond to $k = (2-4n)$ -shifted cases). In fact, for Calabi-Yau $(4n+2)$ -folds (i.e. $k \equiv 0 \pmod{4}$), the determinant line bundle has a canonical trivialization [7]. To explain it in a simple way, we take a simple sheaf \mathcal{F} , the determinant line bundle $\mathcal{L}_{\mathcal{M}_X}$ satisfies

$$\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{F}} \cong \det(Ext^{2n+1}(\mathcal{F}, \mathcal{F})),$$

as other terms are of type $\det(V \oplus V^*)$ and have canonical trivializations. The Serre duality pairing defines a non-degenerate 2-form (instead of a quadratic form) on $Ext^{2n+1}(\mathcal{F}, \mathcal{F})$ (see also Theorem 5.1 at the level of spin representations), which gives a canonical trivialization of $\det(Ext^{2n+1}(\mathcal{F}, \mathcal{F}))$ as in the holomorphic symplectic case (see Mukai [42]).

Remark 2.4. Index bundles could be understood as tangent bundles of moduli spaces in the derived sense. By Theorem 2.2 and the above discussion, we can regard moduli spaces of simple sheaves on Calabi-Yau $2n$ -folds as 'derived' Calabi-Yau spaces.

3. ORIENTABILITY FOR ODD DIMENSIONAL CALABI-YAU

In [43], Nekrasov and Okounkov gave a short proof of the existence of square roots of determinant line bundles for moduli spaces of sheaves on Calabi-Yau $(2n+1)$ -folds. However, sometimes it would be useful to make the square root compatible with other structures, such as the wall-crossing structure in the sense of Kontsevich and Soibelman [35] on moduli spaces. In [28], Hua showed that finding (wall-crossing compatible) square roots is related to find square roots of determinant line bundles over spaces of gauge equivalent classes of connections. Following the argument of Donaldson [21], he then used geometric transitions to prove the existence of square roots for simply-connected torsion-free Calabi-Yau 3-folds.

In this section, we will show that the existence of square roots is in fact a phenomenon in spin geometry, and prove their existence over spaces of gauge equivalent classes of irreducible connections on spin manifolds X 's with $\dim_{\mathbb{R}}(X) = 8k+2$ or $8k+6$.

Theorem 3.1. *Let X be a compact spin manifold of real dimension $8k+2$ or $8k+6$, $(E, h) \rightarrow X$ be a (Hermitian) complex vector bundle, and $N \geq \text{rk}(E) + 1$ be a positive integer.*

Then there exists a $SU(N)$ complex vector bundle E' and a continuous map $s : \tilde{\mathcal{B}}_{E,X} \rightarrow \tilde{\mathcal{B}}_{E',X}$ such that the quotient of determinant line bundles

$$\frac{\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}}))}{s^* \det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}'}))}$$

has a canonical square root.

Furthermore, if $N \gg 0$, $\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}'}))$ has square roots whose choices are parametrized by $\text{Hom}(H^{\text{odd}}(X, \mathbb{Z}), \mathbb{Z}_2)$.

Proof. As in the proof of Theorem 2.1, by considering $E' = E \oplus (\det E)^{-1} \oplus \mathbb{C}^p$, we have a stabilization map

$$s : \tilde{\mathcal{B}}_{E,X} \rightarrow \tilde{\mathcal{B}}_{E',X}, \quad s(A) = A \oplus (\det(A))^* \oplus \theta$$

where θ is the rank p product connection. When a $SU(N)$ connection on E' decomposes as $A \oplus (\det(A))^* \oplus \theta$, there is a decomposition of the endomorphism bundle

$$\text{End}E' = \text{End}E \oplus T^*V \oplus \text{End}(\mathbb{C}^p), \quad V \triangleq ((\det E) \otimes E) \oplus ((\mathbb{C}^p)^* \otimes E) \oplus ((\det E) \otimes \mathbb{C}^p).$$

The index of any operator coupled with this bundle (with connection) is a sum of corresponding terms. In the obvious notation, we have

$$s^*(\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}'}))) = \det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}})) \otimes \det(\text{Ind}(\mathbb{P}_{T^*V})) \otimes \det(\text{Ind}(\mathbb{P}_{\text{End}\mathbb{C}^p})).$$

By Corollary 5.4, $\det(\text{Ind}(\mathbb{P}_{T^*V})) \cong (\det(\text{Ind}(\mathbb{P}_V)))^{\otimes 2}$, which has a canonical square root. $\det(\text{Ind}(\mathbb{P}_{\text{End}\mathbb{C}^p}))$ is canonically trivial as \mathbb{C}^p is a product bundle. Then

$$s^*(\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}'}))) \cong \det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}})) \otimes (\det(\text{Ind}(\mathbb{P}_V)))^{\otimes 2}.$$

Following Nekrasov and Okounkov [43], we consider an involution map

$$\begin{aligned} \sigma : \tilde{\mathcal{B}}_{E',X} \times \tilde{\mathcal{B}}_{E',X} &\rightarrow \tilde{\mathcal{B}}_{E',X} \times \tilde{\mathcal{B}}_{E',X}, \\ \sigma([A_1], [A_2]) &= ([A_2], [A_1]). \end{aligned}$$

We denote the extended determinant line bundle to be $\mathcal{L} = \det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}'})) \rightarrow \tilde{\mathcal{B}}_{E',X} \times \tilde{\mathcal{B}}_{E',X}$. By applying the canonical isomorphism in Theorem 5.3 to $\text{End}E$, we can obtain

$$(2) \quad \sigma^* \mathcal{L} \cong \mathcal{L}.$$

Now we are reduced to prove $c_1(\mathcal{L}|_{\Delta}) \equiv 0 \pmod{2}$, where $\Delta \hookrightarrow \tilde{\mathcal{B}}_{E',X} \times \tilde{\mathcal{B}}_{E',X}$ is the diagonal.

By the Künneth formula,

$$\begin{aligned} H^2(\tilde{\mathcal{B}}_{E',X} \times \tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2) &\cong H^0(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2) \otimes H^2(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2) \oplus \\ &\oplus H^2(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2) \otimes H^0(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2) \oplus H^1(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2) \otimes H^1(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2). \end{aligned}$$

Assume $\{a_i\}$ is a basis of $H^0(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2)$, $\{b_i\}$ is a basis of $H^2(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2)$, $\{c_i\}$ is a basis of $H^1(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}_2)$, and

$$c_1(\mathcal{L}) \equiv \sum_{i,j} n_{ij} a_i \otimes b_j + \sum_{i,j} m_{ij} b_i \otimes a_j + \sum_{i,j} k_{ij} c_i \otimes c_j \pmod{2}.$$

Under the action of the involution map σ ,

$$\sigma^*(c_1(\mathcal{L})) \equiv \sum_{i,j} m_{ij} a_j \otimes b_i + \sum_{i,j} n_{ij} b_j \otimes a_i + \sum_{i,j} k_{ij} c_j \otimes c_i \pmod{2}.$$

By (2), we obtain $m_{ji} \equiv n_{ij} \pmod{2}$, $k_{ji} \equiv k_{ij} \pmod{2}$. When we restrict to the diagonal,

$$c_1(\mathcal{L}|_\Delta) \equiv \sum_{i,j} n_{ij}(a_i \cup b_j + b_j \cup a_i) \equiv 0 \pmod{2}.$$

If $N \gg 0$, we have $H_1(\tilde{\mathcal{B}}_{E',X}, \mathbb{Z}) \cong H^{\text{odd}}(X, \mathbb{Z})$ as showed in Theorem 2.1. Meanwhile, the choice of square roots of any complex line bundle over a space W is parametrized by $H^1(W, \mathbb{Z}_2)$. \square

We fix a Calabi-Yau $(2n+1)$ -fold X , and denote the determinant line bundle of a coarse moduli space \mathcal{M}_X of simple sheaves by $\mathcal{L}_{\mathcal{M}_X}$ with $\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{F}} \cong \det(\text{Ext}^{\text{odd}}(\mathcal{F}, \mathcal{F})) \otimes \det(\text{Ext}^{\text{even}}(\mathcal{F}, \mathcal{F}))^{-1}$.

Theorem 3.2. *Let X be a projective Calabi-Yau $(2n+1)$ -fold, \mathcal{M}_X be a proper coarse moduli scheme of simple sheaves with fixed Chern classes.*

Then there is a 1-1 correspondence between the set of principal \mathbb{Z}_2 -bundles on \mathcal{M}_X and the set of algebraic square roots of $\mathcal{O}_{\mathcal{M}_X^{\text{red}}}$. Moreover, each element in $\text{Hom}(H^{\text{odd}}(X, \mathbb{Z}), \mathbb{Z}_2)$ determines an algebraic square root of $\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{\text{red}}}$ over the reduced scheme $\mathcal{M}_X^{\text{red}}$.

Proof. As in the proof of Theorem 2.2, using Theorem 3.1, each element in $\text{Hom}(H^{\text{odd}}(X, \mathbb{Z}), \mathbb{Z}_2)$ determines a (topological) square root of $\mathcal{L}_{\mathcal{M}_X}$. By Lemma 6.1 [43], $\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{\text{red}}}$ has an algebraic square root as $c_1(\mathcal{L}_{\mathcal{M}_X})$ is even. To show those (topological) square roots are algebraic square roots over $\mathcal{M}_X^{\text{red}}$, we are left to show any principal \mathbb{Z}_2 -bundle is an algebraic square root of $\mathcal{O}_{\mathcal{M}_X^{\text{red}}}$.

From the short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathcal{O}_{\mathcal{M}_X^{\text{red}}}^* \xrightarrow{f \mapsto f^2} \mathcal{O}_{\mathcal{M}_X^{\text{red}}}^* \rightarrow 1,$$

we obtain an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{M}_X^{\text{red}}, \mathbb{Z}_2) &\rightarrow H^0(\mathcal{M}_X^{\text{red}}, \mathcal{O}_{\mathcal{M}_X^{\text{red}}}^*) \xrightarrow{f \mapsto f^2} H^0(\mathcal{M}_X^{\text{red}}, \mathcal{O}_{\mathcal{M}_X^{\text{red}}}^*) \rightarrow \\ &\rightarrow H^1(\mathcal{M}_X^{\text{red}}, \mathbb{Z}_2) \xrightarrow{i} H^1(\mathcal{M}_X^{\text{red}}, \mathcal{O}_{\mathcal{M}_X^{\text{red}}}^*) \xrightarrow{L \mapsto L^2} H^1(\mathcal{M}_X^{\text{red}}, \mathcal{O}_{\mathcal{M}_X^{\text{red}}}^*) \rightarrow \dots \end{aligned}$$

As $\mathcal{M}_X^{\text{red}}$ is proper, any algebraic function on it is locally constant (ref. 10.3.7 of [51]). So the above sequence splits and the map $H^1(\mathcal{M}_X^{\text{red}}, \mathbb{Z}_2) \rightarrow H^1(\mathcal{M}_X^{\text{red}}, \mathcal{O}_{\mathcal{M}_X^{\text{red}}}^*)$ is injective. Using the remaining exact sequence, it is obvious that the corresponding algebraic line bundle (image under map i) is a square root of $\mathcal{O}_{\mathcal{M}_X^{\text{red}}}$. \square

Remark 3.3. Based on Joyce's definition of orientations for d-critical locus [31], we only need square roots of determinant line bundles over the reduced moduli schemes to categorify DT_3 invariants [9].

The following example shows that one could not expect to get vanishing of first Chern classes of determinant line bundles for odd dimensional CY.

Example 3.4. Let X be a generic quintic 3-fold, and consider the Hilbert scheme of two points on X (which is smooth), i.e. $\text{Hilb}^{(2)}(X) = \text{Bl}_\Delta(X \times X)/\mathbb{Z}_2$, where $\Delta \hookrightarrow X \times X$ is the diagonal. Its determinant line bundle satisfies $c_1(\mathcal{L}_{\text{Hilb}^{(2)}(X)}) = 2c_1(\text{Hilb}^{(2)}(X)) \neq 0$.

4. ORIENTABILITY FOR THE RELATIVE CASE

4.1. The weak orientability result. We take a smooth (Calabi-Yau) $(2n-1)$ -fold Y in a smooth complex projective $2n$ -fold X as its anti-canonical divisor. We denote \mathcal{M}_X to be a coarse moduli space of simple bundles on X with fixed Chern classes which has a well-defined restriction morphism

$$r : \mathcal{M}_X \rightarrow \mathcal{M}_Y,$$

to a coarse moduli space of simple bundles on Y with fixed Chern classes.

The corresponding restriction morphism between reduced schemes⁶ is still denoted by

$$r : \mathcal{M}_X^{\text{red}} \rightarrow \mathcal{M}_Y^{\text{red}}.$$

By Theorem 3.2, there exists square roots of $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{\text{red}}}$ coming from the restriction of square roots of the determinant line bundle of the index bundle of twisted Dirac operators over the space of connections. The following relative orientability result for the morphism r gives an identification of complex line bundles $r^*(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{\text{red}}})^{\frac{1}{2}} \cong \mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{\text{red}}}$.

⁶It is uniquely determined by r , see Ex. 2.3 [26].

Theorem 4.1. (*Weak relative orientability*)

Let Y be a smooth anti-canonical divisor in a projective $2n$ -fold X with $\text{Tor}(H_*(X, \mathbb{Z})) = 0$, $E \rightarrow X$ be a complex vector bundle with structure group $SU(N)$, where $N \gg 0$. Let \mathcal{M}_X be a coarse moduli scheme of simple holomorphic structures on E , which has a well-defined restriction morphism

$$r : \mathcal{M}_X \rightarrow \mathcal{M}_Y,$$

to a proper coarse moduli scheme of simple bundles on Y with fixed Chern classes.

Then there exists an algebraic square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{\text{red}}})^{\frac{1}{2}}$ of $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{\text{red}}}$ such that

$$c_1(\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{\text{red}}}) = r^*c_1((\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{\text{red}}})^{\frac{1}{2}}),$$

where $\mathcal{L}_{\mathcal{M}_X}$ (resp. $\mathcal{L}_{\mathcal{M}_Y}$) is the determinant line bundle of \mathcal{M}_X (resp. \mathcal{M}_Y).

Proof. Without loss of generality, we assume \mathcal{M}_X and \mathcal{M}_Y are reduced schemes. Using a Hermitian metric on E , we have embeddings $\mathcal{M}_X \subseteq \mathcal{B}_X^*$, $\mathcal{M}_Y \subseteq \mathcal{B}_Y^*$ into spaces of gauge equivalent classes of irreducible unitary connections on E and $E|_Y$. The restriction map $r : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ extends to a restriction map $r : \mathcal{B}_X^* \rightarrow \mathcal{B}_Y^*$ to the orbit space of connections on $E|_Y$. We consider an open subset $U(\mathcal{M}_X) \triangleq r^{-1}(\mathcal{B}_Y^*)$ of \mathcal{B}_X^* which fits into a commutative diagram

$$(3) \quad \begin{array}{ccc} \beta^{-1}(U(\mathcal{M}_X)) & \xrightarrow{r} & \tilde{\mathcal{B}}_Y^* \\ \beta \downarrow & & \downarrow \beta \\ U(\mathcal{M}_X) & \xrightarrow{r} & \mathcal{B}_Y^*, \end{array}$$

where β is a $PSU(N)$ -fiber bundle defined by forgetting the framing at a base point $x_0 \in Y \subseteq X$ (see also (5.1.3) of [21]).

By Theorem 3.2, there exists an algebraic square root $\mathcal{L}_{\mathcal{M}_Y}^{\frac{1}{2}}$ coming from a square root $\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_Y}))^{\frac{1}{2}} \rightarrow \tilde{\mathcal{B}}_Y^*$. In fact, its restriction to $\tilde{\mathcal{B}}_Y^* \subseteq \tilde{\mathcal{B}}_Y$ descends to a line bundle over \mathcal{B}_Y^* via map β , which gives $\mathcal{L}_{\mathcal{M}_Y}^{\frac{1}{2}}$ on $\mathcal{M}_Y \subseteq \mathcal{B}_Y^*$. Thus to prove the result for Chern classes of line bundles over \mathcal{B}_X^* , we are left to prove a result for $G = PSU(N)$ -equivariant Chern classes over the space $\tilde{\mathcal{B}}_X^*$, i.e.

$$(4) \quad c_1^G(\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_X}))) - c_1^G(r^*\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_Y})))^{\frac{1}{2}} = 0 \in H^2(\tilde{\mathcal{B}}_X^* \times_G EG, \mathbb{Z}),$$

for the $SU(N)$ complex vector bundle $E \rightarrow X$ with $N \gg 0$, where $r : \tilde{\mathcal{B}}_X^* \rightarrow \tilde{\mathcal{B}}_Y^*$ is the restriction map which extends the one in (3), and \mathcal{E}_X (resp. \mathcal{E}_Y) is the universal family over $\tilde{\mathcal{B}}_X^*$ (resp. $\tilde{\mathcal{B}}_Y^*$). The index bundle $\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_X})$ is defined by a lifting $c_1(X)$ of $w_2(X)$ using the spin^c structure of complex manifold X .

The Federer spectral sequence

$$E_2^{p,q} \cong H^p(X, \pi_{p+q}(BSU(N))) \Rightarrow \pi_q(\text{Map}_E(X, BSU(N))).$$

gives $\pi_1(\text{Map}_E(X, BSU(N))) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z})$ for $N \gg 0$, which is torsion-free. Since G acts on $\tilde{\mathcal{B}}_X^* \times EG$ freely, we have an exact sequence

$$\pi_1(G) \rightarrow \pi_1(\tilde{\mathcal{B}}_X^* \times EG) \rightarrow \pi_1(\tilde{\mathcal{B}}_X^* \times_G EG) \rightarrow 0.$$

As $\pi_1(G) \cong \mathbb{Z}_n$, and $\pi_1(\tilde{\mathcal{B}}_X^* \times EG) \cong \pi_1(\tilde{\mathcal{B}}_X^*) \cong \pi_1(\text{Map}_E(X, BSU(N)))$ is torsion-free, there is no homomorphism from torsion groups to torsion-free ones, thus $\pi_1(\tilde{\mathcal{B}}_X^* \times_G EG) \cong \pi_1(\tilde{\mathcal{B}}_X^*)$. And $H_1(\tilde{\mathcal{B}}_X^* \times_G EG, \mathbb{Z})$, $H^2(\tilde{\mathcal{B}}_X^* \times_G EG, \mathbb{Z})$ are torsion-free.

Thus to prove (4), we only need

$$2c_1(\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_X})) \times_G EG) - c_1(r^*\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_Y})) \times_G EG) = 0 \in H^2(\tilde{\mathcal{B}}_X^* \times_G EG, \mathbb{Q}).$$

We are furthermore left to show

$$2c_1(\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_X})) \times_G EG)|_C - c_1(r^*\det(\text{Ind}(\mathbb{P}_{\text{End}\mathcal{E}_Y})) \times_G EG)|_C = 0 \in H^2(C, \mathbb{Q})$$

for any embedded surface $C \subseteq \tilde{\mathcal{B}}_X^* \times_G EG$.

Note that the universal bundle \mathcal{E}_X is G -invariant and extends to a universal bundle $\mathcal{E}_X \times_G EG$ over $\tilde{\mathcal{B}}_X^* \times_G EG$. We denote the universal bundle over C to be $\mathcal{E} \rightarrow X \times C$, $\pi_X : X \times C \rightarrow C$,

$\pi_Y : Y \times C \rightarrow C$ to be projection maps, and $i = (i_Y, Id) : Y \times C \rightarrow X \times C$. The commutative diagram

$$\begin{array}{ccc} Y \times C & \xrightarrow{i} & X \times C \\ & \searrow \pi_Y & \swarrow \pi_X \\ & C & \end{array}$$

implies that $\pi_{X!} \circ i_! = \pi_{Y!}$ for Gysin homomorphisms on cohomologies. By applying the Atiyah-Singer family index theorem [4], we have

$$\begin{aligned} c_1(\det(\text{Ind}(\mathbb{D}_{\text{End}\mathcal{E}_X})) \times_G EG)|_C &= c_1(\det(\text{Ind}(\mathbb{D}_{\text{End}(\mathcal{E}_X \times_G EG)}))|_C) \\ &= \pi_{X!}([ch(\text{End}\mathcal{E}) \cdot Td(X)]^{(2n+1)}) \\ &= \left(\sum_{i=1}^n ch_{2i}(\text{End}\mathcal{E}) \cdot Td_{2n-2i+1}(X)\right)/[X], \\ c_1(r^* \det(\text{Ind}(\mathbb{D}_{\text{End}\mathcal{E}_Y})) \times_G EG)|_C &= c_1(r^* \det(\text{Ind}(\mathbb{D}_{\text{End}(\mathcal{E}_Y \times_G EG)}))|_C) \\ &= \pi_{Y!}([ch(\text{End}(i^*\mathcal{E})) \cdot Td(Y)]^{(2n)}) \\ &= \pi_{X!} \circ i_!([i^* ch(\text{End}\mathcal{E}) \cdot \frac{Td(X)|_Y}{Td(\mathcal{N}_{Y/X})}]^{(2n)}) \\ &= \pi_{X!}([i_! \circ i^*(ch(\text{End}\mathcal{E}) \cdot \frac{Td(X)}{Td(K_X^{-1})})]^{(2n+1)}) \\ &= \pi_{X!}([ch(\text{End}\mathcal{E}) \cdot \frac{Td(X)}{Td(K_X^{-1})} \cdot c_1(K_X^{-1})]^{(2n+1)}) \\ &= [ch(\text{End}\mathcal{E}) \cdot Td(X) \cdot (1 - e^{-c_1(X)})]^{(2n+1)}/[X]. \end{aligned}$$

We introduce $\widetilde{Td}(X) = Td(X) \cdot (1 - e^{-c_1(X)})$. To prove

$$2c_1(\det(\text{Ind}(\mathbb{D}_{\text{End}\mathcal{E}_X})) \times_G EG)|_C = c_1(r^* \det(\text{Ind}(\mathbb{D}_{\text{End}\mathcal{E}_Y})) \times_G EG)|_C,$$

we are left to show

$$(5) \quad \widetilde{Td}_{2i-1}(X) = 2 Td_{2i-1}(X), \text{ for } 1 \leq i \leq n,$$

i.e. $\widetilde{Td}(X) - 2 Td(X)$ consists of even index classes. Note that the \hat{A} -class satisfies

$$Td(X) = e^{\frac{c_1(X)}{2}} \cdot \hat{A}(X),$$

and

$$\begin{aligned} \widetilde{Td}(X) - 2Td(X) &= \hat{A}(X)(e^{\frac{c_1(X)}{2}} - e^{-\frac{c_1(X)}{2}}) - 2\hat{A}(X) \cdot e^{\frac{c_1(X)}{2}} \\ &= -\hat{A}(X)(e^{\frac{c_1(X)}{2}} + e^{-\frac{c_1(X)}{2}}), \end{aligned}$$

which is of even index as both factors in the RHS are so. \square

4.2. Relations with relative orientations for restriction morphisms. We start with the restriction morphism $r : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ between two coarse moduli spaces and determinant line bundles $\mathcal{L}_{\mathcal{M}_X}, \mathcal{L}_{\mathcal{M}_Y}$ over them respectively.

Lemma 4.2. *There exists a canonical isomorphism*

$$\alpha : (\mathcal{L}_{\mathcal{M}_X})^{\otimes 2} \cong r^* \mathcal{L}_{\mathcal{M}_Y}$$

between algebraic line bundles.

Proof. We consider a commutative diagram

$$\begin{array}{ccc} Y \times \mathcal{M}_X & \xrightarrow{i} & X \times \mathcal{M}_X \\ & \searrow \pi_Y & \swarrow \pi_X \\ & \mathcal{M}_X & \end{array}$$

By definition, $r^* \mathcal{L}_{\mathcal{M}_Y} = \det((R\pi_Y)_*(R\mathcal{H}om(i^*\mathcal{E}, i^*\mathcal{E})))$, where $\mathcal{E} \rightarrow X \times \mathcal{M}_X$ is the universal bundle of \mathcal{M}_X . By the adjunction formula (see for instance [29]), we have

$$\begin{aligned} (R\pi_Y)_*(R\mathcal{H}om(i^*\mathcal{E}, i^*\mathcal{E})) &= (R\pi_X)_* \circ i_*(i^* R\mathcal{H}om(\mathcal{E}, \mathcal{E})) \\ &= (R\pi_X)_*(R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes^L \mathcal{O}_{Y \times \mathcal{M}_X}). \end{aligned}$$

From the short exact sequence $0 \rightarrow p^*K_X \rightarrow \mathcal{O}_{X \times \mathcal{M}_X} \rightarrow \mathcal{O}_{Y \times \mathcal{M}_X} \rightarrow 0$, where $p : X \times \mathcal{M}_X \rightarrow X$ is the projection, we obtain exact triangles

$$(6) \quad \begin{aligned} R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes p^*K_X &\rightarrow R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \rightarrow R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes^L \mathcal{O}_{Y \times \mathcal{M}_X}, \\ (R\pi_X)_*(R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes p^*K_X) &\rightarrow (R\pi_X)_*R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \rightarrow (R\pi_X)_*(R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes^L \mathcal{O}_{Y \times \mathcal{M}_X}). \end{aligned}$$

By the Grothendieck-Serre duality [16], we have

$$(7) \quad (R\pi_X)_*(R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes p^*K_X[2n]) \simeq ((R\pi_X)_*R\mathcal{H}om(\mathcal{E}, \mathcal{E}))^\vee.$$

By taking the corresponding element of (6) in the Grothendieck group (see for instance page 124 of [29]), and using (7), we can obtain an isomorphism of determinant line bundles

$$(\mathcal{L}_{\mathcal{M}_X})^{\otimes 2} \cong r^*\mathcal{L}_{\mathcal{M}_Y}.$$

□

Remark 4.3.

1. When $\dim_{\mathbb{C}} X = 2$ and Y is an elliptic curve, \mathcal{M}_X is smooth by Serre duality. The above isomorphism gives $(K_{\mathcal{M}_X})^{\otimes 2} \cong \mathcal{O}_{\mathcal{M}_X}$. This fits into the work of Bottacin [8] on the existence of Poisson structures on moduli spaces of simple sheaves on Poisson surfaces. By Proposition 6.2 [8], the Poisson structure on \mathcal{M}_X is non-degenerate and defines a holomorphic symplectic structure if r is a well-defined map to a coarse moduli of simple bundles on Y .
2. Similar isomorphism holds true for Lagrangians in $(3-2n)$ -shifted symplectic derived schemes in the sense of Pantev, Töen, Vaquié and Vezzosi [46].

Then it is natural to make the following definition for orientations in this relative set-up (compare to Definition 2.11 of Borisov and Joyce [7]).

Definition 4.4. Let X be a smooth projective $2n$ -fold with a smooth anti-canonical divisor $Y \in |K_X^{-1}|$, and $r : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ be a well-defined restriction morphism between coarse moduli spaces of simple sheaves on X and Y with fixed Chern classes respectively.

A *relative orientation* for morphism r consists of a square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$ of the determinant line bundle $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}$ and an isomorphism

$$\theta : \mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}} \cong r^*(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$$

such that $\theta \otimes \theta \cong \alpha$ holds over \mathcal{M}_X^{red} for the isomorphism α in Lemma 4.2.

Remark 4.5.

1. When X is a CY_{2n} , $Y = \emptyset$, the above definition coincides with Definition 2.11 of [7] on orientations of even-shifted symplectic derived schemes, because a real line bundle is trivial on a scheme if and only if it is trivial over its reduced scheme (see also Theorem 2.2).
2. When $\dim_{\mathbb{C}} X = 4$, relative orientations are required to defined relative DT_4 invariants [15].

The previous Theorem 4.1 gives an evidence for the existence of relative orientations. Another partial result is given as follows.

Proposition 4.6. *We assume $H^1(\mathcal{M}_X, \mathbb{Z}_2) = 0$. Then relative orientations for restriction morphism $r : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ exist.*

Proof. We denote the uniquely determined restriction morphism between reduced schemes also by $r : \mathcal{M}_X^{red} \rightarrow \mathcal{M}_Y^{red}$. Combining with Lemma 4.2, there exists a canonical isomorphism

$$\alpha : (\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}})^{\otimes 2} \cong r^*(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}).$$

We take any square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$ of $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}$ (existence is due to [43]). Thus α determines a trivialization

$$\alpha : ((\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}}) \otimes r^*(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{-\frac{1}{2}})^{\otimes 2} \cong \mathcal{O}_{\mathcal{M}_X^{red}}.$$

From the short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathcal{O}_{\mathcal{M}_X^{red}}^* \xrightarrow{f \mapsto f^2} \mathcal{O}_{\mathcal{M}_X^{red}}^* \rightarrow 1,$$

we obtain an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{M}_X^{red}, \mathbb{Z}_2) &\rightarrow H^0(\mathcal{M}_X^{red}, \mathcal{O}_{\mathcal{M}_X^{red}}^*) \xrightarrow{f \mapsto f^2} H^0(\mathcal{M}_X^{red}, \mathcal{O}_{\mathcal{M}_X^{red}}^*) \rightarrow \\ &\rightarrow H^1(\mathcal{M}_X^{red}, \mathbb{Z}_2) \xrightarrow{i} H^1(\mathcal{M}_X^{red}, \mathcal{O}_{\mathcal{M}_X^{red}}^*) \xrightarrow{L \mapsto L^2} H^1(\mathcal{M}_X^{red}, \mathcal{O}_{\mathcal{M}_X^{red}}^*) \rightarrow \dots \end{aligned}$$

$H^1(\mathcal{M}_X, \mathbb{Z}_2) = 0$ implies that any square root of $\mathcal{O}_{\mathcal{M}_X^{red}}$ is $\mathcal{O}_{\mathcal{M}_X^{red}}$. \square

5. APPENDIX

5.1. Some basic facts in spin geometry. In this section, we recall some basic facts in spin geometry. The main references are the book [36] of Lawson and Michelson and the book [1] by Adams.

Theorem 5.1. (*Theorem 4.6 [1]*)

Let $\Delta_{\mathbb{C}}^{\pm}$ be two fundamental complex spinor representations of $Spin(n)$, where n is even.

Then, we have

- (1) $\Delta_{\mathbb{C}}^{\pm}$ are real if $n = 8k$;
- (2) $\Delta_{\mathbb{C}}^{\pm}$ are symplectic if $n = 8k + 4$;
- (3) $\Delta_{\mathbb{C}}^+ \cong (\Delta_{\mathbb{C}}^-)^*$ if $n = 8k + 2$ or $8k + 6$.

Remark 5.2. In case (1), $\Delta_{\mathbb{C}}^{\pm}$ are endowed with $Spin(n)$ -invariant non-degenerate quadratic forms as they are complexifications of real representations.

In case (2), $\Delta_{\mathbb{C}}^{\pm}$ are endowed with $Spin(n)$ -invariant non-degenerate 2-forms.

Let $(X, g, P_{Spin}(TX))$ be an even dimensional spin manifold. We define its complex spinor bundles by

$$\mathcal{S}_{\mathbb{C}}^{\pm}(X) = P_{Spin}(TX) \times_{Spin(n)} \Delta_{\mathbb{C}}^{\pm}.$$

As $\mathcal{S}_{\mathbb{C}}^{\pm}(X)$ are Clifford bundles, and the Levi-Civita connection on (X, g) induces connections ∇ 's on complex vector bundles $\mathcal{S}_{\mathbb{C}}^{\pm}(X)$, there exists Dirac operators

$$\mathcal{D}^{\pm} : \Gamma(\mathcal{S}_{\mathbb{C}}^{\pm}(X)) \rightarrow \Gamma(\mathcal{S}_{\mathbb{C}}^{\mp}(X)).$$

If $\dim_{\mathbb{R}}(X) = 8k$, by Theorem 5.1, $\mathcal{S}_{\mathbb{C}}^{\pm}(X)$ are the complexification of real spinor bundles $\mathcal{S}^{\pm}(X)$. The corresponding Dirac operators (with their kernel and cokernel) are the complexifications of the real ones.

If $\dim_{\mathbb{R}}(X) = 8k + 2$ or $8k + 6$, by Theorem 5.1, two complex spinor bundles $\mathcal{S}_{\mathbb{C}}^{\pm}(X)$ are dual to each other as bundles with left $Cl(X)$ -action, i.e. there is an isomorphism

$$(8) \quad \mathcal{S}_{\mathbb{C}}^+(X) \cong (\mathcal{S}_{\mathbb{C}}^-(X))^*,$$

which is equivariant under the action of Clifford bundle $Cl(X)$.

Let $(E, h) \rightarrow X$ be a Hermitian complex vector bundle, and A be an unitary connection. We can define twisted Dirac operators

$$\mathcal{D}_A^+ : \Gamma(\mathcal{S}_{\mathbb{C}}^+(X) \otimes E) \rightarrow \Gamma(\mathcal{S}_{\mathbb{C}}^-(X) \otimes E),$$

$$\mathcal{D}_{A^*}^- : \Gamma(\mathcal{S}_{\mathbb{C}}^-(X) \otimes E^*) \rightarrow \Gamma(\mathcal{S}_{\mathbb{C}}^+(X) \otimes E^*).$$

As $\mathcal{S}_{\mathbb{C}}^+(X)$ and $\mathcal{S}_{\mathbb{C}}^-(X)$ are dual as Clifford bundles, when they are coupled with dual bundles E and E^* , kernels of the corresponding twisted Dirac operators are dual to each other, i.e.

Theorem 5.3. *Let X be a compact spin manifold with $\dim_{\mathbb{R}}(X) = 8k + 2$ or $8k + 6$, and $E \rightarrow X$ be a (Hermitian) complex vector bundle with an unitary connection A . Then there is a canonical isomorphism*

$$\ker(\mathcal{D}_A^+) \cong (\ker(\mathcal{D}_{A^*}^-))^*$$

of complex vector spaces.

As a direct corollary of Theorem 5.3, we have

Corollary 5.4. *Let X be a compact spin manifold with $\dim_{\mathbb{R}}(X) = 8k + 2$ or $8k + 6$, and (E, h) be a (Hermitian) complex vector bundle with an unitary connection A . Then for Dirac operator*

$$\mathcal{D}_{A \oplus A^*}^+ : \Gamma(\mathcal{S}_{\mathbb{C}}^+(X) \otimes T^*E) \rightarrow \Gamma(\mathcal{S}_{\mathbb{C}}^-(X) \otimes T^*E),$$

we have a canonical isomorphism

$$\det(\text{Ind}(\mathcal{D}_{A \oplus A^*}^+)) \cong (\det(\text{Ind}(\mathcal{D}_A^+)))^{\otimes 2}$$

for determinant lines.

Remark 5.5. The above canonical isomorphisms naturally extend to corresponding determinant line bundles.

5.2. Some standard material from gauge theory. In this subsection, we recall some standard material from gauge theory. The main references are the book [21] by Donaldson and Kronheimer and a series of papers [17], [18], [19] by Donaldson.

We fix a compact spin manifold X of even dimension and a Hermitian complex vector bundle $(E, h) \rightarrow X$. Given an unitary connection A on E , we can define the twisted Dirac operator

$$\not{D}_{A^* \otimes A} : \Gamma(\not{S}_{\mathbb{C}}^+(X) \otimes \text{End} E) \rightarrow \Gamma(\not{S}_{\mathbb{C}}^-(X) \otimes \text{End} E)$$

following Theorem 13.10 of [36]. $[\ker(\not{D}_{A^* \otimes A}) - \text{coker}(\not{D}_{A^* \otimes A})]$ exists as an element in the K -theory $K(pt)$ of one point, and there is a family version of the above construction as follows.

Let \mathcal{A} be the space of all unitary connections on (E, h) , and \mathcal{G} be the group of unitary gauge transformations. If we denote the $U(r)$ -principal bundle of E by P ,

$$\mathcal{G} = \Gamma(X, P \times_{U(r)} U(r))$$

is the space of C^∞ -sections of the bundle $P \times_{U(r)} U(r)$ (with the conjugate action $U(r) \curvearrowright U(r)$). The orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ (with suitable Sobolev structure) exists as a metrizable topological space (Lemma 4.2.4 [21]). Following [21], we fix a base point $x_0 \in X$ and introduce the space

$$\tilde{\mathcal{B}}_X = \mathcal{A} \times_{\mathcal{G}} P_{x_0}$$

of equivalent classes of framed connections. Equivalently, $\tilde{\mathcal{B}}_X = \mathcal{A}/\mathcal{G}_0$, where $\mathcal{G}_0 \triangleleft \mathcal{G}$ is the subgroup of gauge transformations which fix the fiber P_{x_0} . As \mathcal{G}_0 acts on \mathcal{A} freely, $\tilde{\mathcal{B}}_X$ (with suitable Sobolev structure) has a Banach manifold structure whose weak homotopy type will not depend on the chosen Sobolev structures (Proposition 5.1.4 [21]). Meanwhile, there exists a universal bundle $\mathcal{E} = \mathcal{A} \times_{\mathcal{G}_0} E$ over $\tilde{\mathcal{B}}_X \times X$ (trivialized on $\tilde{\mathcal{B}}_X \times \{x_0\}$), which carries a universal family of framed connections.

We then couple the Dirac operator \not{D} on X with the universal connection on \mathcal{E} and there is an index bundle

$$\text{Ind}(\not{D}_{\text{End} \mathcal{E}}) \in K(\tilde{\mathcal{B}}_X),$$

which satisfies $\text{Ind}(\not{D}_{\text{End} \mathcal{E}})|_{[A]} = \ker(\not{D}_{A^* \otimes A}) - \text{coker}(\not{D}_{A^* \otimes A})$ [4] (see also page 181 of [21]). For any finite dimensional submanifold $C \subseteq \tilde{\mathcal{B}}_X$ and the induced family $\mathbb{E} = \mathcal{E}|_C$ over $C \times X$, we have the Atiyah-Singer family index formula [4]:

$$\text{ch}(\text{Ind}(\not{D}_{\text{End} \mathbb{E}})) = (\text{ch}(\text{End} \mathbb{E}) \cdot \hat{A}(X))/[X].$$

5.3. Seidel-Thomas twists. In this section, we recall the Seidel-Thomas twist [47] and how it could be used to identify a moduli space of simple sheaves to a moduli space of simple holomorphic bundles, which is the work of Joyce and Song [33].

Definition 5.6. Let $(X, \mathcal{O}_X(1))$ be a projective Calabi-Yau m -fold with $\text{Hol}(X) = SU(m)$. For each $n \in \mathbb{Z}$, the Seidel-Thomas twist $T_{\mathcal{O}_X(-n)}$ by $\mathcal{O}_X(-n)$ is the Fourier-Mukai transform from $D(X)$ to $D(X)$ with kernel

$$K = \text{cone}(\mathcal{O}_X(n) \boxtimes \mathcal{O}_X(-n) \rightarrow \mathcal{O}_\Delta).$$

In general, $T_n \triangleq T_{\mathcal{O}_X(-n)}[-1]$ maps sheaves to complexes of sheaves. But for $n \gg 0$, we have

Theorem 5.7. (Joyce-Song, Lemma 8.2 of [33]) *Let U be a finite type \mathbb{C} -scheme and \mathcal{F}_U is a coherent sheaf on $U \times X$ flat over U i.e. it is a U -family of coherent sheaves on X . Then for $n \gg 0$, $T_n(\mathcal{F}_U)$ is also a U -family of coherent sheaves on X .*

Sufficiently many compositions of Seidel-Thomas twists map sheaves to vector bundles.

Definition 5.8. For a nonzero coherent sheaf \mathcal{F} , the homological dimension $\text{hd}(\mathcal{F})$ is the smallest $n \geq 0$ for which there exists an exact sequence in the abelian category $\text{coh}(X)$ of coherent sheaves

$$0 \rightarrow E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_0 \rightarrow \mathcal{F} \rightarrow 0$$

with $\{E_i\}_{i=0, \dots, n}$ are vector bundles.

Theorem 5.9. (Joyce-Song, Lemma 8.4 of [33]) *Let \mathcal{F}_U , $n \gg 0$ be the same as in Theorem 5.7, then for any $u \in U$, we have $\text{hd}(T_n(\mathcal{F}_u)) = \max(\text{hd}(\mathcal{F}_u) - 1, 0)$.*

Corollary 5.10. (Joyce-Song, Corollary 8.5 of [33]) *Let U be a finite type \mathbb{C} -scheme and \mathcal{F}_U is a U -family of coherent sheaves on X . Then there exists $n_1, \dots, n_m \gg 0$ such that for $T_{n_m} \circ \cdots \circ T_{n_1}(\mathcal{F}_U)$ is a U -family of vector bundles on X .*

Meanwhile, Seidel-Thomas twists are auto-equivalences of derived category $D(X)$, they preserve determinant line bundles (if exists) of corresponding moduli spaces.

Corollary 5.11. (*Joyce-Song [33]*) *Given a coarse moduli space \mathcal{M}_X of simple sheaves with fixed Chern classes, we choose sufficiently large integers $n_1, \dots, n_m \gg 0$ such that $\Psi \triangleq T_{n_m} \circ \dots \circ T_{n_1}$ identifies \mathcal{M}_X with a coarse moduli space $\mathfrak{M}_X^{\text{bdl}}$ of simple holomorphic bundles. Then*

$$\Psi^* \mathcal{L}_{\mathfrak{M}_X^{\text{bdl}}} \cong \mathcal{L}_{\mathcal{M}_X},$$

where \mathcal{L}_{\bullet} is the determinant line bundle of the corresponding moduli space.

Moreover, if X is a CY_{2n} , $\mathcal{L}_{\mathfrak{M}_X^{\text{bdl}}}$ and $\mathcal{L}_{\mathcal{M}_X}$ are endowed with non-degenerate quadratic forms from Serre duality pairing. The isomorphism $\Psi^* \mathcal{L}_{\mathfrak{M}_X^{\text{bdl}}} \cong \mathcal{L}_{\mathcal{M}_X}$ also preserves the quadratic forms.

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